AIAA JOURNAL
Vol. 45, No. 6, June 2007

Novel Modal Method for Efficient Calculation of Complex Eigenvector Derivatives

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DOI: 10.2514/1.28035

A novel modal superposition method is presented for calculating eigenvector derivatives in self-adjoint or non-self-adjoint systems. The method is especially appropriate for determination of derivatives of many eigenvectors and requires only one eigenvalue and its corresponding right and left eigenvectors. Two parameters for the eigenvalue under consideration are introduced to obtain a generic derivative formula, which contains most of the available modal superposition methods as a branch. Matrices of the modified system are decomposed with available lower modes to avoid repeatedly solving equations for different eigenvalues. The contribution of truncated higher modes is expanded in a continued product of a few matrices, polynomials, and one generalized power series, whereby rapid convergence can be achieved easily, even in the vicinity of the convergent boundary, without additional computation cost. All other modal superposition coefficients can be neglected intrinsically with the determination of introduced parameters, but a simple condition for coefficients associated with lower-order eigenvalues should be satisfied in a special case. Numerical examples show that the method is efficient and can give results comparable to the exact solutions. The method is applicable to various damped systems and closed-mode cases.

Nomenclature

$\boldsymbol{A}$, $\boldsymbol{B}$	=	coefficient matrices of the eigenproblem
$\boldsymbol{B}_1$	=	replacement of matrix $\boldsymbol{B}$
d	=	shift value for zero eigenvalues
$\boldsymbol{M}$, $\boldsymbol{K}$, $\boldsymbol{C}$, $\boldsymbol{D}$	=	mass, stiffness, viscous damping, and hysteretic damping matrices, only used in numerical examples
m_f	=	number of power polynomials
N	=	order of the eigenequation
N_{tr}	=	number of modes used for modal superposition
n_i	=	number of terms of the ith power polynomial
q	=	exponential of matrix
$\boldsymbol{x}_q$, $\bar{\boldsymbol{x}}_q$	=	solutions of variational systems, corresponding to right and left eigenvectors, respectively
x_ε	=	positive root
α_{rk}, $\bar{\alpha}_{\text{rk}}$	=	modal superposition coefficients of right and left eigenvector derivatives, respectively
δ	=	relative error in eigenvector derivative
$\partial/\partial p$	=	partial derivative with respect to parameter p
$\varepsilon(x)$	=	relative error function
λ_r	=	rth eigenvalue
υ, μ	=	parameters associated with the rth eigenvalue
$\boldsymbol{\phi}_r$, $\bar{\boldsymbol{\phi}}_r$	=	rth right and left eigenvectors, respectively
$\boldsymbol{\Phi}$, $\bar{\boldsymbol{\Phi}}$	=	right and left eigenvector matrices, respectively
$\boldsymbol{\Lambda}$	=	diagonal matrix of eigenvalues

Superscripts

T	=	transpose of a matrix
-1	=	inverse of a matrix

Subscripts

H, L	=	quantities associated with higher and lower modes, respectively
RS, LS	=	quantities associated with right eigenspace and left eigenspace, respectively

I. Introduction

THE eigenvector derivatives with respect to system parameters are extensively used in stability analysis, health monitoring, finite element method (FEM) model updating, dynamic structural control and optimization, and so on. At the present time, research on real eigenvector derivatives is almost mature [1–4]. For complex eigenvector derivative problems, however, many problems still remain to be solved, especially for accuracy and efficiency.

Methods for directly solving the singular control equations have been paid great attention since the earlier work of Fox and Kapoor [1]. These methods could be clearly distinguished into the direct method [4–6], the modal method [7–9], and the iteration method [10]. The direct methods are exact analytical methods. The methods modify the singular system matrix to be nonsingular with normalization conditions and then solve the modified algebraic equations to obtain eigenvector derivatives. Nelson's [4] method requires knowledge of only one eigenvalue and its associated right and left eigenvectors and is recommended as a superior method [7]. Recently, Friswell and Adhikari [6] extended Nelson's method to nonproportionally damped systems. However, these mentioned direct methods are less efficient because they repeatedly solve equations for different eigenvalues when derivatives of many eigenvectors are demanded. To address the problem, Zhang and Wei [11] proposed a series of dynamic flexibility methods based on the theory of practical complete modal space. The basic idea of their methods includes perturbation to the original governing equations and generalized inverse technique [12]. Recent progress improves the convergent rate in closed-mode cases by using two shifting frequency values [13]. So far, their efforts focus on the undamped system. In this case, the modal method may be more competent.

The modal method calculates the eigenvector derivatives by expanding them in terms of eigenvectors. The use of a complete set of eigenvectors will result in an exact solution [1]. Practically, it is difficult to acquire all eigenvectors for a real system, so then subset of them is used and, consequently, leads to heavy truncated error. To

Received 2 September 2006; accepted for publication 16 January 2007.

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improve accuracy and speed up convergence, Wang [8] first employed a pseudo-static response term in the solution. The sequent improvement included modal acceleration to approximate the contribution from the truncated higher modes [9]. A similar result was obtained by Akgun [14] for non-self-adjoint systems. The idea of shifting frequency was incorporated to improve convergence further in the works of Ma and Hagiwara [15] and Zeng [16] for real modes and viscously damped systems, respectively. However, no common rule exists for finding a fixed shift value for all eigenvectors in the case of complex modes. Moreover, a shifting frequency approach would be more effective only near a specific eigenvalue. The accuracy and efficiency are still insufficient when many eigenvector derivatives are required.

In this paper, a generic modal superposition formula is first presented by introducing two parameters for the eigenvalue under consideration; most of the current modal methods belong to a branch of it. When many derivatives of eigenvectors are of interest, by decomposing system matrices only once, an algorithm is developed with available lower modes, to expand matrices similar to dynamic flexibility, and rapid convergence can be achieved easily, even in the vicinity of the convergent boundary, whereas its operation count is far less than that of the power series used in [17]. Relations between the accuracy of the expansion expression and the introduced parameters are established. The conditions to neglect the other modal superposition coefficients are found to be simple and can be satisfied easily. Consequently, only one eigenvalue and its corresponding right and left eigenvectors are actually needed. Two numerical examples demonstrate the efficiency and validity of the presented method. The first one is a non-self-adjoint system with zero eigenvalues. The second one is a self-adjoint system with nonproportional damping and hysteretic damping. When many derivatives of eigenvectors are of interest, the method may be the most efficient and accurate among the current modal methods, and is more efficient than Nelson's [4] method, because no repeatedly solving algebraic equations is needed. The method is applicable to various damping types and is appropriate for the closed-mode case, in either a self-adjoint system or a non-self-adjoint system.

II. Fundamentals

Consider the eigenequation of the form

$$(\lambda_r \boldsymbol{A} + \boldsymbol{B})\boldsymbol{\phi}_r = \boldsymbol{0} \tag{1}$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are nonsymmetric $N \times N$ matrices. The corresponding left eigenvector is given by the following equation:

$$\bar{\boldsymbol{\phi}}_r^T(\lambda_r \boldsymbol{A} + \boldsymbol{B}) = \boldsymbol{0}^T \tag{2a}$$

or

$$(\lambda_r \boldsymbol{A}^T + \boldsymbol{B}^T)\bar{\boldsymbol{\phi}}_r = \boldsymbol{0} \tag{2b}$$

when the system is self-adjoint, that is, $\boldsymbol{A} = \boldsymbol{A}^T$ and $\boldsymbol{B} = \boldsymbol{B}^T$, the left and right eigenvectors are equal.

Assuming that the eigenvalues are distinct, then orthonormal conditions may be expressed as follows:

$$\bar{\boldsymbol{\Phi}}^T \boldsymbol{A} \boldsymbol{\Phi} = \boldsymbol{I} \tag{3a}$$

$$\bar{\boldsymbol{\Phi}}^T \boldsymbol{B} \boldsymbol{\Phi} = -\boldsymbol{\Lambda} \tag{3b}$$

$$\boldsymbol{\phi}_r^T \boldsymbol{A} \boldsymbol{\phi}_r = 1 \qquad r = 1, 2, \ldots, N \tag{3c}$$

$$\boldsymbol{\phi}_r^T \boldsymbol{B} \boldsymbol{\phi}_r = -\lambda_r \qquad r = 1, 2, \ldots, N \tag{3d}$$

where $\boldsymbol{I}$ is the identity matrix; $\bar{\boldsymbol{\Phi}}$ and $\boldsymbol{\Phi}$ are $N \times N$ invertible matrices for which the columns are left and right eigenvectors, respectively; and for $\boldsymbol{\Lambda}$, the entry in the rth row is the eigenvalue associated with the eigenvector in the rth column of $\bar{\boldsymbol{\Phi}}$ and $\boldsymbol{\Phi}$. The normalization procedure is arranged according to Eq. (3c) for $\boldsymbol{\phi}_r$ and Eq. (3a) for $\bar{\boldsymbol{\phi}}_r$, consequently, Eqs. (3b) and (3d) are satisfied naturally.

If $\boldsymbol{B}$ is singular, that is, zero eigenvalues exist in the system, Eqs. (1) and (2) should first be converted to $(\rho_r \boldsymbol{A} + \boldsymbol{B}_1)\boldsymbol{\phi}_r = \boldsymbol{0}$ and $\bar{\boldsymbol{\phi}}_r^T(\rho_r \boldsymbol{A} + \boldsymbol{B}_1) = \boldsymbol{0}^T$ before performance of derivatives, whereby $\boldsymbol{B}_1 = \boldsymbol{B} - d\boldsymbol{A}$ is nonsingular, $\rho_r = \lambda_r + d$, and d is a real or complex constant. Except for a constant difference of d between the original eigenvalues and the new eigenvalues of the modified system, the eigenvectors and various derivatives remained unchanged. Therefore, derivatives can be calculated in the modified system.

A. Governing Equations

Taking a derivative of Eqs. (1–3) with respect to system parameter p and using orthonormal conditions (3a), (3b), and (3d) gives

$$(\lambda_r \boldsymbol{A} + \boldsymbol{B})\frac{\partial \boldsymbol{\phi}_r}{\partial p} = \boldsymbol{g}_r \boldsymbol{\phi}_r \tag{4a}$$

$$(\lambda_r \boldsymbol{A}^T + \boldsymbol{B}^T)\frac{\partial \bar{\boldsymbol{\phi}}_r}{\partial p} = \boldsymbol{g}_r^T \bar{\boldsymbol{\phi}}_r \tag{4b}$$

$$\frac{\partial \bar{\boldsymbol{\phi}}_r^T}{\partial p}\boldsymbol{A}\boldsymbol{\phi}_r + \bar{\boldsymbol{\phi}}_r^T \boldsymbol{A}\frac{\partial \boldsymbol{\phi}_r}{\partial p} = -\bar{\boldsymbol{\phi}}_r^T \frac{\partial \boldsymbol{A}}{\partial p}\boldsymbol{\phi}_r \tag{4c}$$

$$\frac{\partial \bar{\boldsymbol{\phi}}_r^T}{\partial p}\boldsymbol{B}\boldsymbol{\phi}_r + \bar{\boldsymbol{\phi}}_r^T \boldsymbol{B}\frac{\partial \boldsymbol{\phi}_r}{\partial p} = \lambda_r \bar{\boldsymbol{\phi}}_r^T \frac{\partial \boldsymbol{A}}{\partial p}\boldsymbol{\phi}_r \tag{4d}$$

$$\boldsymbol{\phi}_r^T(\boldsymbol{A} + \boldsymbol{A}^T)\frac{\partial \boldsymbol{\phi}_r}{\partial p} = -\boldsymbol{\phi}_r^T \frac{\partial \boldsymbol{A}}{\partial p}\boldsymbol{\phi}_r \tag{4e}$$

$$\boldsymbol{\phi}_r^T(\boldsymbol{B} + \boldsymbol{B}^T)\frac{\partial \boldsymbol{\phi}_r}{\partial p} = \lambda_r \bar{\boldsymbol{\phi}}_r^T \frac{\partial \boldsymbol{A}}{\partial p}\boldsymbol{\phi}_r + (\bar{\boldsymbol{\phi}}_r - \boldsymbol{\phi}_r)^T \frac{\partial \boldsymbol{B}}{\partial p}\boldsymbol{\phi}_r \tag{4f}$$

where

$$\boldsymbol{g}_r = -\left(\frac{\partial \lambda_r}{\partial p}\boldsymbol{A} + \lambda_r \frac{\partial \boldsymbol{A}}{\partial p} + \frac{\partial \boldsymbol{B}}{\partial p}\right) \tag{5}$$

$$\frac{\partial \lambda_r}{\partial p} = -\bar{\boldsymbol{\phi}}_r^T\left(\lambda_r \frac{\partial \boldsymbol{A}}{\partial p} + \frac{\partial \boldsymbol{B}}{\partial p}\right)\boldsymbol{\phi}_r \tag{6}$$

With the assumption of distinct eigenvalues, coefficient matrices of Eqs. (4a) and (4b) are of the rank $N - 1$, and so Eqs. (4a–4f) will be used as supplementation.

B. Solution for Eigenvector Derivatives by Introducing Two Parameters

Two unknown parameters (υ and μ, associated with the rth mode) are introduced herein, which can be real or complex and will be determined in a later section. Adding and subtracting $\upsilon \boldsymbol{A} + \mu \boldsymbol{B}$ in parentheses on the left-hand side of Eq. (4a), gives

$$\{(\mu \boldsymbol{B} + \upsilon \boldsymbol{A}) - [(\mu - 1)\boldsymbol{B} + (\upsilon - \lambda_r)\boldsymbol{A}]\}\frac{\partial \phi_r}{\partial p} = \boldsymbol{g}_r \boldsymbol{\phi}_r \tag{7}$$

If $\upsilon/\mu \neq \lambda_k (k = 1, 2, ..., N)$ then $(\upsilon \boldsymbol{A} + \mu \boldsymbol{B})$ is invertible and Eq. (7) can be rewritten as

$$(\boldsymbol{I} - \boldsymbol{W})(\mu \boldsymbol{B} + \upsilon \boldsymbol{A})\frac{\partial \boldsymbol{\phi}_r}{\partial p} = \boldsymbol{g}_r \boldsymbol{\phi}_r$$

where $\boldsymbol{W} = [(\mu - 1)\boldsymbol{B} + (\upsilon - \lambda_r)\boldsymbol{A}](\mu \boldsymbol{B} + \upsilon \boldsymbol{A})^{-1}$. Premultiplying the preceding equation by $\boldsymbol{W}^{k-1}$, then summing the q equations with k varying from one to q, gives

$$(\boldsymbol{I} - \boldsymbol{W}^q)(\mu \boldsymbol{B} + \upsilon \boldsymbol{A})\frac{\partial \boldsymbol{\phi}_r}{\partial p} = \sum_{k=1}^{q} \boldsymbol{W}^{k-1} \boldsymbol{g}_r \boldsymbol{\phi}_r$$

Regarding $\upsilon \boldsymbol{A} + \mu \boldsymbol{B}$ as a new system, $\boldsymbol{x}_q$ as the response of the system, and the right-hand side of the preceding equation as exciting force, one can immediately obtain

$$(\mu \boldsymbol{B} + \upsilon \boldsymbol{A})\boldsymbol{x}_q = (\boldsymbol{I} - \boldsymbol{W}^q)(\mu \boldsymbol{B} + \upsilon \boldsymbol{A})\frac{\partial \boldsymbol{\phi}_r}{\partial p} = \sum_{k=1}^{q} \boldsymbol{W}^{k-1}\boldsymbol{g}_r \boldsymbol{\phi}_r \quad (8)$$

From the preceding equations, one knows that $\boldsymbol{x}_q$ always exists, that is,

$$\boldsymbol{x}_q = (\mu \boldsymbol{B} + \upsilon \boldsymbol{A})^{-1} \sum_{k=1}^{q} \boldsymbol{W}^{k-1}\boldsymbol{g}_r \boldsymbol{\phi}_r \quad (9)$$

and the relationship between $\boldsymbol{x}_q$ and $\partial \boldsymbol{\phi}_r / \partial p$ is clear, that is,

$$\frac{\partial \boldsymbol{\phi}_r}{\partial p} - \boldsymbol{x}_q = (\mu \boldsymbol{B} + \upsilon \boldsymbol{A})^{-1}\boldsymbol{W}^q(\mu \boldsymbol{B} + \upsilon \boldsymbol{A})\frac{\partial \boldsymbol{\phi}_r}{\partial p}$$

If $\partial \boldsymbol{\phi}_r / \partial p - \boldsymbol{x}_q$ is already known, then $\partial \boldsymbol{\phi}_r / \partial p$ can be readily calculated. Here, the modal superposition method will be employed. First, premultiplying Eq. (9) by $(\lambda_r \boldsymbol{A} + \boldsymbol{B})$ then subtracting the result from Eq. (4a) using Eqs. (3a) and (3b) and the invertibility of $\bar{\boldsymbol{\Phi}}$ and $\boldsymbol{\Phi}$ gives

$$(\lambda_r \boldsymbol{A} + \boldsymbol{B})\left(\frac{\partial \boldsymbol{\phi}_r}{\partial p} - \boldsymbol{x}_q\right) = \bar{\boldsymbol{\Phi}}^{-T}\{[(1-\mu)\boldsymbol{\Lambda} + (\upsilon - \lambda_r)\boldsymbol{I}](\upsilon \boldsymbol{I} - \mu \boldsymbol{\Lambda})^{-1}\}^q \bar{\boldsymbol{\Phi}}^T \boldsymbol{g}_r \boldsymbol{\phi}_r \quad (10)$$

then, the exact solution of the preceding equation can be expressed as the sum of all right eigenvectors:

$$\frac{\partial \boldsymbol{\phi}_r}{\partial p} - \boldsymbol{x}_q = \sum_{k=1}^{N} \alpha_{\mathrm{rk}} \boldsymbol{\phi}_k \quad (11)$$

In practice, only part of the lower modes are available, then

$$\frac{\partial \boldsymbol{\phi}_r}{\partial p} = \boldsymbol{x}_q + \sum_{k=1}^{N_{tr}} \alpha_{\mathrm{rk}} \boldsymbol{\phi}_k \quad (12)$$

To determine coefficients α_{rk}, substituting Eq. (11) into Eq. (10) and premultiplying Eq. (10) by $\bar{\boldsymbol{\phi}}_k^T$ gives

$$\alpha_{\mathrm{rk}} = \left(\frac{(1-\mu)\lambda_k + \upsilon - \lambda_r}{\upsilon - \mu \lambda_k}\right)^q \frac{\bar{\boldsymbol{\phi}}_k^T\left(\lambda_r \frac{\partial \boldsymbol{A}}{\partial p} + \frac{\partial \boldsymbol{B}}{\partial p}\right)\boldsymbol{\phi}_r}{\lambda_k - \lambda_r} \qquad k \neq r \quad (13)$$

In an analogous way, according to the procedures from Eqs. (7–12) and considering Eq. (4b), the following results can be obtained for left eigenvectors:

$$\bar{\boldsymbol{x}}_q = (\mu \boldsymbol{B}^T + \upsilon \boldsymbol{A}^T)^{-1} \sum_{k=1}^{q} \{[(\mu - 1)\boldsymbol{B}^T + (\upsilon - \lambda_r)\boldsymbol{A}^T] \times (\mu \boldsymbol{B}^T + \upsilon \boldsymbol{A}^T)^{-1}\}^{k-1} \boldsymbol{g}_r^T \bar{\boldsymbol{\phi}}_r \quad (14)$$

$$\frac{\partial \bar{\boldsymbol{\phi}}_r}{\partial p} = \bar{\boldsymbol{x}}_q + \sum_{k=1}^{N_{tr}} \bar{\alpha}_{\mathrm{rk}} \bar{\boldsymbol{\phi}}_k \quad (15)$$

$$\bar{\boldsymbol{\alpha}}_{\mathrm{rk}} = \left(\frac{(1-\mu)\lambda_k + \upsilon - \lambda_r}{\upsilon - \mu \lambda_k}\right)^q \frac{\boldsymbol{\phi}_k^T\left(\lambda_r \frac{\partial \boldsymbol{A}^T}{\partial p} + \frac{\partial \boldsymbol{B}^T}{\partial p}\right)\bar{\boldsymbol{\phi}}_r}{\lambda_k - \lambda_r} \qquad k \neq r \quad (16)$$

To determine α_{rr} and $\bar{\alpha}_{rr}$, two equations are needed, which rely on supplementary conditions (4c–4f). Multiplying Eq. (4c) by $(\upsilon - \lambda_r)$ and Eq. (4d) by $(\mu - 1)$, noting that they are scalar equations, then transposing the first term in each equation and summing the two equations gives

$$\boldsymbol{\phi}_r^T[(\mu - 1)\boldsymbol{B}^T + (\upsilon - \lambda_r)\boldsymbol{A}^T]\frac{\partial \bar{\boldsymbol{\phi}}_r}{\partial p} + \bar{\boldsymbol{\phi}}_r^T[(\mu - 1)\boldsymbol{B} + (\upsilon - \lambda_r)\boldsymbol{A}]\frac{\partial \boldsymbol{\phi}_r}{\partial p} = [(\mu - 1)\lambda_r - (\upsilon - \lambda_r)]\bar{\boldsymbol{\phi}}_r^T \frac{\partial \boldsymbol{A}}{\partial p}\boldsymbol{\phi}_r \quad (17)$$

With eigenequation (1), the following recurrence relations can be obtained:

$$\boldsymbol{\phi}_r^T = \boldsymbol{\phi}_r^T\{[(\mu - 1)\boldsymbol{B}^T + (\upsilon - \lambda_r)\boldsymbol{A}^T](\mu \boldsymbol{B}^T + \upsilon \boldsymbol{A}^T)^{-1}\}^k$$

Combining Eq. (4b), one can prove that

$$\boldsymbol{\phi}_r^T[(\mu - 1)\boldsymbol{B}^T + (\upsilon - \lambda_r)\boldsymbol{A}^T]\bar{\boldsymbol{x}}_q = 0 \quad (18a)$$

Similarly, with Eqs. (2a) and (4a), the following identity is also obtained:

$$\bar{\boldsymbol{\phi}}_r^T[(\mu - 1)\boldsymbol{B} + (\upsilon - \lambda_r)\boldsymbol{A}]\boldsymbol{x}_q = 0 \quad (18b)$$

Substituting Eqs. (12) and (15) into Eq. (17) and noting Eqs. (18a) and (18b) gives

$$\alpha_{rr} + \bar{\alpha}_{rr} = -\bar{\boldsymbol{\phi}}_r^T \frac{\partial \boldsymbol{A}}{\partial p}\boldsymbol{\phi}_r \quad (19)$$

Repeating the procedures from Eqs. (17–19), replacing Eq. (4c) with Eq. (4e) and Eq. (4d) with Eq. (4f), noting no use of Eqs. (15), (18a), and (18b), results in

$$\alpha_{rr} = -\frac{1}{2} \cdot \frac{1}{(1-\mu)\lambda_r + \upsilon - \lambda_r}\left\{[(1-\mu)\lambda_r \bar{\boldsymbol{\phi}}_r + (\upsilon - \lambda_r)\boldsymbol{\phi}_r]^T \frac{\partial \boldsymbol{A}}{\partial p}\boldsymbol{\phi}_r - (\mu - 1)(\bar{\boldsymbol{\phi}}_r - \boldsymbol{\phi}_r)^T \frac{\partial \boldsymbol{B}}{\partial p}\boldsymbol{\phi}_r + \boldsymbol{\phi}_r^T[(\mu - 1)(\boldsymbol{B} + \boldsymbol{B}^T) + (\upsilon - \lambda_r)(\boldsymbol{A} + \boldsymbol{A}^T)] \times \left(\boldsymbol{x}_q + \sum_{k=1, k\neq r}^{N_{tr}} \alpha_{\mathrm{rk}} \boldsymbol{\phi}_k\right)\right\} \quad (20)$$

subsequently, $\bar{\alpha}_{rr}$ can be calculated from Eq. (19).

For a self-adjoint system, Eqs. (18a) or (18b) can be used, and with the orthonormal conditions (3a) and (3b), the preceding equation is simplified to

$$\alpha_{rr} = -\frac{1}{2}\boldsymbol{\phi}_r^T \frac{\partial \boldsymbol{A}}{\partial p}\boldsymbol{\phi}_r \quad (21)$$

Very basic formulas are obtained for calculation of eigenvector derivatives in self-adjoint or non-self-adjoint systems, except the introduced parameters υ and μ still remain to be solved. The method is generic. Most of the current available modal superposition methods would be directly obtained by letting $\mu = 1$ and taking different values of υ and q. Therefore, these methods belong to a branch of the presented method. Relationships between these methods will be discussed in a later section.

An alternative determination of υ and μ will be given in successive sections, whereby any α_{rk} and $\bar{\alpha}_{\mathrm{rk}} (k \neq r)$ could be discarded and only α_{rr} and $\bar{\alpha}_{rr}$ are retained. In addition, the matrix $(\upsilon \boldsymbol{A} + \mu \boldsymbol{B})^{-1}$ in Eq. (9) or $(\upsilon \boldsymbol{A}^T + \mu \boldsymbol{B}^T)^{-1}$ in Eq. (14) always needs to be calculated for different eigenvalues. However, the repeated use of the direct inverse approach is less efficient and so should not be employed in this case. Quick calculation of $(\upsilon \boldsymbol{A} + \mu \boldsymbol{B})^{-1}$ will be completed in the next section.

C. Decomposition of the Inverse Matrix of $(\upsilon \boldsymbol{A} + \mu \boldsymbol{B})$

Here, calculation of the inverse matrix is converted to a series of multiplication operations of matrices and available lower modes are used. A similar idea could be found in [12,13,17], in which a conventional generalized power series was used for a real mode case. However, the conventional power series will lead to large operation counts and low accuracy. The developed algorithm herein

emphasizes a few multiplication operations, rapid convergence, and capability of accuracy-holding within a large scope.

The right eigenvector matrix $\boldsymbol{\Phi}$ is first partitioned into two submatrices $\boldsymbol{\Phi}_L$ and $\boldsymbol{\Phi}_H$. Columns of $\boldsymbol{\Phi}_L$ include all available lower eigenvectors, and columns of $\boldsymbol{\Phi}_H$ are unknown or truncated. Partition of the left eigenvector matrix $\bar{\boldsymbol{\Phi}}$ is in the same way. Correspondingly, matrix $\boldsymbol{\Lambda}$ becomes $\boldsymbol{\Lambda}_L$, $\boldsymbol{\Lambda}_H$, and two zero submatrices at opposite locations along the diagonal. From orthonormal conditions (3b), using the invertibility of $\bar{\boldsymbol{\Phi}}$ and $\boldsymbol{\Phi}$ and noting the aforementioned partitions of matrices, one can obtain

$$\boldsymbol{B}^{-1} = -(\boldsymbol{R}_L + \boldsymbol{R}_H) \tag{22}$$

where $\boldsymbol{R}_L = \boldsymbol{\Phi}_L \boldsymbol{\Lambda}_L^{-1} \bar{\boldsymbol{\Phi}}_L^T$ and $\boldsymbol{R}_H = \boldsymbol{\Phi}_H \boldsymbol{\Lambda}_H^{-1} \bar{\boldsymbol{\Phi}}_H^T = -\boldsymbol{B}^{-1} - \boldsymbol{\Phi}_L \boldsymbol{\Lambda}_L^{-1} \bar{\boldsymbol{\Phi}}_L^T$.

Postmultiplying $\boldsymbol{R}_H = \boldsymbol{\Phi}_H \boldsymbol{\Lambda}_H^{-1} \bar{\boldsymbol{\Phi}}_H^T$ by $\boldsymbol{R}_H \boldsymbol{A}$ using orthonormal conditions (3a) and repeating this process gives

$$\boldsymbol{\Phi}_H \boldsymbol{\Lambda}_H^{-k} \bar{\boldsymbol{\Phi}}_H^T = (\boldsymbol{R}_H \boldsymbol{A})^{k-1} \boldsymbol{R} \tag{23}$$

Matrix $(\upsilon \boldsymbol{A} + \mu \boldsymbol{B})^{-1}$ can now be rewritten as

$$(\upsilon \boldsymbol{A} + \mu \boldsymbol{B})^{-1} = \boldsymbol{\Phi}_L [\upsilon \boldsymbol{I}_L - \mu \boldsymbol{\Lambda}_L]^{-1} \bar{\boldsymbol{\Phi}}_L^T + \boldsymbol{\Phi}_H [\upsilon \boldsymbol{I}_H - \mu \boldsymbol{\Lambda}_H]^{-1} \bar{\boldsymbol{\Phi}}_H^T \tag{24}$$

The first term on right-hand side of Eq. (24) can be directly calculated with available lower modes. However, the second term is unknown, which represents contribution from truncated higher modes and needs to be determined with known information. The summation form of the second term is

$$\boldsymbol{\Phi}_H [\upsilon \boldsymbol{I}_H - \mu \boldsymbol{\Lambda}_H]^{-1} \bar{\boldsymbol{\Phi}}_H^T = -\frac{1}{\mu} \sum_{h=l+1}^{N} \frac{\boldsymbol{\phi}_h \bar{\boldsymbol{\phi}}_h^T}{\lambda_h [1 - (\upsilon/\mu\lambda_h)]} \tag{25}$$

To expand $1/[1 - \upsilon/(\mu\lambda_h)]$, the following function is used

$$f(x) = \frac{1}{1-x} = \prod_{i=1}^{m_f} \sum_{j_i=0}^{n_i} x^{j_i \prod_{k=1}^{i} (n_{k-1}+1)} \qquad (n_0 = 0, |x| < 1) \tag{26}$$

The first $m_f - 1$ factors are power polynomials, respectively, and the m_fth factor is the $(n_{m_f} + 1)$th partial sum of a power series. The last factor converges rapidly because of the quickly increasing exponentials of its terms. Thereby, convergence of the right-hand side of Eq. (26) would be sped up. A conventional power series can be obtained if we let $m_f = 1$. Furthermore, the right-hand side of Eq. (26) also represents a power polynomial with the highest exponential of

$$\sum_{i=1}^{m_f} \left(n_i \prod_{k=1}^{i} (n_{k-1} + 1) \right)$$

if we fully expand it. The strategy is employed during calculation that there are no multiplication operations for $j_i = 0$ or 1, and beginning with the first factor, the highest exponential of each factor (i.e., n_i) increases by one to be the initial value of the next factor. For the same value of $f(x)$,

$$\sum_{i=1}^{m_f} \left(n_i \prod_{k=1}^{i} (n_{k-1} + 1) \right) - 2$$

multiplications are needed for a conventional power series, whereas

$$m_f - 2 + \sum_{i=1}^{m_f} n_i$$

for Eq. (26). On the other hand, for the given multiplication operations, the exponentials of the corresponding fully expanded power series of $f(x)$ can change as m_f and n_i vary and need not explicitly give the series. Obviously, Eq. (26) would benefit the closed-mode case, in which the error of conventional power series worsens rapidly as x moves away from zero, and simply augmenting the number of terms in the series would not effectively improve convergence because of the drastic increase of operation counts. Another idea for overcoming the shortage of conventional power series is to scale x to be small, so that no additional computation cost arises, even in the vicinity of the convergent boundary. This idea will be implemented with the determination of υ and μ in the next section.

The relative error of Eq. (26) is

$$\begin{aligned} \varepsilon(x) &= 1 - (1-x) \prod_{i=1}^{m_f} \sum_{j_i=0}^{n_i} x^{j_i \prod_{k=1}^{i} (n_{k-1}+1)} \\ &= x^{1 + \sum_{i=1}^{m_f} [n_i \prod_{k=1}^{i} (n_{k-1}+1)]} \end{aligned} \tag{27}$$

and if $x \in [0, 1)$, then $\varepsilon(x)$ is a monotone increasing function. This character will be used when determining parameters υ and μ in the next section.

Regarding $\upsilon/(\mu\lambda_h)$ as x, substituting Eq. (26) into Eq. (25) using orthonormal condition (3a) and noting Eq. (23), after a series of matrix operations, the following result is obtained:

$$\begin{aligned} &\boldsymbol{\Phi}_H [\upsilon \boldsymbol{I}_H - \mu \boldsymbol{\Lambda}_H]^{-1} \bar{\boldsymbol{\Phi}}_H^T \\ &\quad = -\frac{1}{\mu} \left[\prod_{i=1}^{m_f} \sum_{j_i=0}^{n_i} \left(\frac{\upsilon}{\mu} \boldsymbol{R}_H \boldsymbol{A} \right)^{j_i \prod_{k=1}^{i} (n_{k-1}+1)} \right] \boldsymbol{R}_H \end{aligned} \tag{28}$$

and then

$$\begin{aligned} (\upsilon \boldsymbol{A} + \mu \boldsymbol{B})^{-1} &= \boldsymbol{\Phi}_L [\upsilon \boldsymbol{I}_L - \mu \boldsymbol{\Lambda}_L]^{-1} \bar{\boldsymbol{\Phi}}_L^T \\ &\quad - \frac{1}{\mu} \left[\prod_{i=1}^{m_f} \sum_{j_i=0}^{n_i} \left(\frac{\upsilon}{\mu} \boldsymbol{R}_H \boldsymbol{A} \right)^{j_i \prod_{k=1}^{i} (n_{k-1}+1)} \right] \boldsymbol{R}_H \end{aligned} \tag{29}$$

With the knowledge of $(\upsilon \boldsymbol{A} + \mu \boldsymbol{B})^{-1}$, the matrix $(\upsilon \boldsymbol{A}^T + \mu \boldsymbol{B}^T)^{-1}$ can be easily obtained as

$$(\upsilon \boldsymbol{A}^T + \mu \boldsymbol{B}^T)^{-1} = [(\upsilon \boldsymbol{A} + \mu \boldsymbol{B})^{-1}]^T \tag{30}$$

With Eq. (29), calculation of $\boldsymbol{B}^{-1}$ is needed only once for all υ and μ when many eigenvector derivatives are of interest. Calculating $\boldsymbol{B}^{-1}$ could be regarded as calculating static response with N unit load cases, and triangular factorization scheme combining with N backward substitution operations would be efficient.

D. Determination of υ and μ

Two independent equations are required to determine two unknown parameters. Let us first consider the relative error formula Eq. (27). For previously chosen m_f and n_i, a root $x_\varepsilon \in (0, 1)$ must exist for a predetermined $\varepsilon \in (0, 1)$. In addition, the necessary condition that Eq. (25) can be expanded as Eq. (28) is $|\upsilon/(\mu\lambda_h)| < 1$ for all h. Then, for the rth mode, let

$$\upsilon/\mu = x_\varepsilon \lambda_r \tag{31}$$

so that $|\upsilon/(\mu\lambda_h)| = |x_\varepsilon \lambda_r/\lambda_h|$. Because $r < h$, so $|\lambda_r/\lambda_h| \le 1$, and consequently $|x_\varepsilon \lambda_r/\lambda_h| \le x_\varepsilon < 1$. Note that $|x_\varepsilon \lambda_r/\lambda_h|$ decreases as h increases; as a result, the right-hand side of Eq. (28) becomes more accurate, because $\varepsilon(x)$ is a monotone increasing function. The obvious advantage of using Eq. (31) is that the convergent rate of Eq. (28) is controlled by x_ε regardless of the value of λ_r. Once m_f and n_i in Eq. (26) are chosen for predetermined ε, they will never change with different λ_r. This means no additional computing cost and benefits the closed-mode case in which λ_r is close to λ_h.

To obtain another equation, let us consider the factor with a power of q of coefficient α_{rk} or $\bar{\alpha}_{\mathrm{rk}}$:

$$\left(\frac{(1-\mu)\lambda_k + \upsilon - \lambda_r}{\upsilon - \mu\lambda_k} \right)^q \tag{32}$$

Note that υ and μ are arbitrary, and so it is reasonable to let the combination of them be

$$\upsilon - \mu\lambda_k = \gamma - \lambda_k + \lambda_r \tag{33}$$

where γ is also an unknown parameter. Recall that υ and μ are associated with the rth mode, they have no relation with other modes. Hence, items that are dependent of arbitrary kth mode, $k \neq r$, should be eliminated from Eq. (33). To do so, let

$$\gamma = -\mu\lambda_k + \lambda_k \tag{34}$$

subsequently, substituting Eq. (34) into Eq. (33) and combining Eq. (31) gives

$$\upsilon = \lambda_r, \qquad \mu = \frac{1}{x_\varepsilon} \tag{35}$$

Now all of the requirements for calculating the eigenvector derivatives are obtained.

Such determined υ and μ makes coefficients α_{rk} or $\bar{\alpha}_{\mathrm{rk}}$ possess some unique characters that are absent for conventional modal superposition methods. With Eq. (35), the factor of α_{rk} or $\bar{\alpha}_{\mathrm{rk}}$ [i.e., Eq. (32)] becomes

$$\left(\frac{(1-x_\varepsilon)\lambda_k}{\lambda_k - x_\varepsilon\lambda_r}\right)^q \tag{36}$$

According to the following basic inequality of complex number,

$$||z_1| - |z_2|| \leq |z_1 - z_2| \tag{37}$$

when $k > r$, it can be easily proven that the following inequality is identically satisfied

$$(1-x_\varepsilon)|\lambda_k| < |\lambda_k - x_\varepsilon\lambda_r|$$

then

$$\left|\frac{(1-x_\varepsilon)\lambda_k}{(\lambda_k - x_\varepsilon\lambda_r)}\right|^q < 1 \tag{38a}$$

and the left-hand side becomes far less than one with increase of q.

Let us consider the case of $k < r$. Denote $\lambda_r = |\lambda_r|e^{i\varphi_r}$, let $\lambda_k = \beta|\lambda_r|e^{i\varphi_k}$, where i is complex identity, β is the ratio of modulus, and $0 < \beta \leq 1$. With substitution of them into Eq. (36) and modulus of the result, inequality (38a) can be proven to be still tenable; nevertheless, the following condition should be satisfied in the case of $\varphi_r - \varphi_k = 2n\pi$ (n is an integer):

$$x_\varepsilon > \frac{2\beta}{1+\beta} \qquad (\beta \neq 1) \tag{38b}$$

or, in its equivalent form,

$$x_\varepsilon > \frac{2|\lambda_k|}{|\lambda_r| + |\lambda_k|} \qquad (|\lambda_r| \neq |\lambda_k|) \tag{38c}$$

The right-hand side of inequality (38c) increases with the increase of $|\lambda_k|$, and so λ_{k-1} certainly satisfies inequality (38c), if λ_k does. The condition that $|\lambda_r| \neq |\lambda_k|$ for $\varphi_r - \varphi_k = 2n\pi$ is naturally satisfied for a system with distinct values.

For any $q \neq 0$, it can be known from Eq. (36) that α_{rk} and $\bar{\alpha}_{\mathrm{rk}}$ approach zero as x_ε approaches one. This indicates that x_ε is the intrinsic cause to determine the property of α_{rk} and $\bar{\alpha}_{\mathrm{rk}}$, and it also reveals the fact that an arbitrary eigenvector derivative has relation mainly to its own eigenmode. From inequality (38), one knows that the moduli of α_{rk} and $\bar{\alpha}_{\mathrm{rk}}$ ($k \neq r$) exponentially decrease. After q is up to a certain number, α_{rk} and $\bar{\alpha}_{\mathrm{rk}}$ ($k \neq r$) can be removed from Eqs. (12), (15), and (20); consequently, only α_{rr} and $\bar{\alpha}_{rr}$ are retained.

Now consider the relationship between the kth and $(k+j)$th modes when $k > r$, where $j = 1, 2, \ldots, N-k$. Note that the following inequality is always tenable:

$$\frac{|\lambda_k|}{|\lambda_{k+j}|} \geq \frac{|\lambda_k| - x_\varepsilon|\lambda_r|}{|\lambda_{k+j}| - x_\varepsilon|\lambda_r|} \geq \frac{|\lambda_k| - x_\varepsilon|\lambda_r|}{|\lambda_{k+j} - x_\varepsilon\lambda_r|}$$

and then

$$1 \geq \frac{(1-x_\varepsilon)|\lambda_k|}{|\lambda_k| - x_\varepsilon|\lambda_r|} \geq \left|\frac{(1-x_\varepsilon)\lambda_{k+j}}{\lambda_{k+j} - x_\varepsilon\lambda_r}\right| \tag{39}$$

This inequality will be used to estimate error.

E. Discussion

1. Calculation Consideration

According to the preceding descriptions of the properties of α_{rk} and $\bar{\alpha}_{\mathrm{rk}}$, x_ε is expected to be close to one for the reduction of the need for large q. However, small x_ε is needed for improvement of the convergent rate of Eq. (28). Hence, x_ε should be compromised between the two requirements. Moreover, x_ε cannot take values independently if Eq. (29) is used. It is obtained through solving Eq. (27) for given m_f, n_i, and predetermined ε; ε controls the accuracy of Eq. (29) and sequentially affects that of eigenvector derivatives. Excessively mismatched x_ε and ε would result in divergence. To obtain larger x_ε with ε unchanged, slightly larger m_f and even distribution of n_i is appropriate if $m_f \neq 1$. Table 1 lists values of x_ε for different m_f and n_i, for 19 multiplication operations and $\varepsilon = 10^{-16}$.

In addition, if lower α_{rk} and $\bar{\alpha}_{\mathrm{rk}}$ are also neglected, inequality (38c) should be examined under the condition of $\varphi_r - \varphi_k = 2n\pi$. It is easy to judge whether $\varphi_r - \varphi_k$ equals $2n\pi$ through sequential comparisons of the signs and ratios of real and imaginary parts of the two eigenvalues. For convenient purpose, inequality (38c) is always calculated before the judgment. With k varying from $r-1$ to 1, if this inequality is already satisfied for a certain k, the rest of the eigenvalues need not be checked any more. Otherwise, the condition $\varphi_r - \varphi_k = 2n\pi$ should be judged. If $\varphi_r - \varphi_k \neq 2n\pi$, the next eigenvalue should be examined. If $\varphi_r - \varphi_k = 2n\pi$, there exists two situations, one is to change x_ε to satisfy the inequality and then stop the examination, the other one is to not neglect the corresponding α_{rk} and $\bar{\alpha}_{\mathrm{rk}}$ and then repeat the examination procedure for the next eigenvalue.

The value of q can be determined as

$$q = \sum_{i=1}^{m_f} n_i \tag{40}$$

Usually, such a value is already large enough in the case of $m_f \neq 1$. If $m_f = 1$, larger q may be needed for convergence.

2. Calculation with Individual Mode

When only derivatives of a specific mode are to be determined, the direct inverse approach could be used to calculate the inverse matrix of $(\upsilon\boldsymbol{A} + \mu\boldsymbol{B})$. Here x_ε can directly take an arbitrary value approximately equal to one. However, if this mode is the unique eigendata, the result may risk invalidation because of the absence of adequate information to verify the possible violation of inequality (38c). An increase of x_ε is helpful for reducing the possibility of invalidation.

3. Consideration on Singularity

As mentioned in the beginning of this paper, the singular $\boldsymbol{B}$ is replaced by $\boldsymbol{B}_1 = \boldsymbol{B} - d\boldsymbol{A}$ to eliminate the singularity of the system before the performance of derivatives, then all eigenvalues become $\lambda + d$. Special attention should be paid to the orders of eigenvalues.

Table 1 Effect of m_f and n_i on x_ε

m_f	n_i				x_ε	ε	Multiplications
	n_1	n_2	n_3	n_4			
3	10	5	3	-	0.870094		
	6	6	3	2	0.939267	10^{-16}	19
4	5	5	5	2	0.944732		
	5	5	4	3	0.950119		

Table 2 Relationship between the presented method and other modal methods

Method	μ	υ	q	System type
Fox and Kapoor [1]	1	0	0	undamped
Sutter et al. [7]	1	0	1	undamped
Wang [9]	1	0	1	undamped
Ma and Hagiwara [15]	1	$\neq 0$ (fixed)	1	undamped
Zhang et al. [3]	1	$\neq 0$ (fixed)	>1	undamped
Akgun [14]	1	0	>1	non-self-adjoint
Zeng [16]	1	$\neq 0$ (fixed, arithmetic mean)	>1	viscously damped
Presented	$1/x_\varepsilon$	λ_r (variable)	>1	self-adjoint or non-self-adjoint

The calculation of $(\upsilon \boldsymbol{A} + \mu \boldsymbol{B}_1)^{-1}$ requires the available lower modes of the original eigensystem. If d changes the lower mode set, that is, orders of some unknown higher eigenvalues become under the available lower eigenvalues, and vice versa, $(\upsilon \boldsymbol{A} + \mu \boldsymbol{B}_1)^{-1}$ will not be calculated correctly. Hence, d should be appropriately selected to avoid the potential change of the lower mode set of the original eigensystem.

4. *Error Estimation for Truncated Modes*

To estimate the effect of ignoring the other modes, the norms in eigenspace spanned by left and right eigenvectors should first be defined. For the right eigenspace, define

$$||\boldsymbol{x}||_{\mathrm{RS}} = ||\bar{\boldsymbol{\Phi}}^T \boldsymbol{A}\boldsymbol{x}||_2 = ||\boldsymbol{S}_1 \boldsymbol{x}||_2 \tag{41}$$

and for the left eigenspace,

$$||\bar{\boldsymbol{x}}||_{\mathrm{LS}} = ||\boldsymbol{\Phi}^T \boldsymbol{A}^T \bar{\boldsymbol{x}}||_2 = ||\boldsymbol{S}_2 \bar{\boldsymbol{x}}||_2 \tag{42}$$

where $||\ ||_2$ denotes 2-norm in unitary space. The norms defined by Eqs. (41) and (42) are helpful to separate α_{rk} and $\bar{\alpha}_{\mathrm{rk}}$ from the linear combination for eigenvector derivative, which is convenient for the derivation of error expression.

Note that Eq. (11) represents the exact solution $\partial \boldsymbol{\phi}_{rE}/\partial p$ of the derivative of the rth right eigenvector, and Eq. (12) denotes the approximate solution. When all of the α_{rk} $(k \neq r)$ are neglected in Eqs. (12) and (20), the norm of the error between $\partial \boldsymbol{\phi}_{rE}/\partial p$ and $\partial \boldsymbol{\phi}_r/\partial p$ can be written as follows with Eq. (41):

$$\left|\left|\frac{\partial \boldsymbol{\phi}_{rE}}{\partial p} - \frac{\partial \boldsymbol{\phi}_r}{\partial p}\right|\right|_{\mathrm{RS}} = \left|\left| \sum_{k=1,k\neq r}^{N} \alpha_{rk}\boldsymbol{\phi}_k + \frac{1}{2\lambda_r}\boldsymbol{\phi}_r^T(\boldsymbol{B} + \boldsymbol{B}^T)\left(\sum_{k=1,k\neq r}^{N} \alpha_{\mathrm{rk}}\boldsymbol{\phi}_k\right)\boldsymbol{\phi}_r \right|\right|_{\mathrm{RS}} \leq \max(c_{rr}, c_{rM}) \left|\left|\frac{\partial \boldsymbol{\phi}_r}{\partial p}\right|\right|_{\mathrm{RS}} \tag{43}$$

where

$$c_{rr} = \left|\frac{1}{\lambda_r}\boldsymbol{\phi}_r^T(\boldsymbol{B} + \boldsymbol{B}^T)\left(\sum_{k=1,k\neq r}^{N} \alpha_{\mathrm{rk}}\boldsymbol{\phi}_k\right) \Big/ \left(\boldsymbol{\phi}_r^T \frac{\partial A}{\partial p}\boldsymbol{\phi}_r + \boldsymbol{\phi}_r^T(\boldsymbol{A} + \boldsymbol{A}^T)\sum_{k=1,k\neq r}^{N} \alpha_{\mathrm{rk}}\boldsymbol{\phi}_k\right)\right|$$

$$c_{rM} = \max\left(\left|\frac{(1-x_\varepsilon)\lambda_1}{\lambda_1 - x_\varepsilon\lambda_r}\right|^q, \ldots, \left|\frac{(1-x_\varepsilon)\lambda_{r-1}}{\lambda_{r-1} - x_\varepsilon\lambda_r}\right|^q, \left|\frac{(1-x_\varepsilon)\lambda_{r+1}}{\lambda_{r+1} - x_\varepsilon\lambda_r}\right|^q, \ldots, \left|\frac{(1-x_\varepsilon)\lambda_N}{\lambda_N - x_\varepsilon\lambda_r}\right|^q\right)$$

Using inequality (39), c_{rM} reduces to

$$c_{rM} = \max\left(\left|\frac{(1-x_\varepsilon)\lambda_1}{\lambda_1 - x_\varepsilon\lambda_r}\right|^q, \ldots, \left|\frac{(1-x_\varepsilon)\lambda_{r-1}}{\lambda_{r-1} - x_\varepsilon\lambda_r}\right|^q, \left(\frac{(1-x_\varepsilon)|\lambda_{r+1}|}{|\lambda_{r+1}| - |x_\varepsilon\lambda_r|}\right)^q\right)$$

where $|\lambda_{r+1}| \neq |\lambda_r|$ because $c_{rM} = 1$ is useless for error estimation. Then the relative error is

$$\delta = \left|\left|\frac{\partial \boldsymbol{\phi}_{rE}}{\partial p} - \frac{\partial \boldsymbol{\phi}_r}{\partial p}\right|\right|_{\mathrm{RS}} \Big/ \left|\left|\frac{\partial \boldsymbol{\phi}_r}{\partial p}\right|\right|_{\mathrm{RS}} \leq \max(c_{rr}, c_{rM}) \tag{44}$$

Similar derivation gives the same result for $\partial \bar{\boldsymbol{\phi}}_r/\partial p$.

Although Eq. (44) does indicate that the error converges to zero, it is impractical for a non-self-adjoint system, because c_{rr} requires all the higher modes. For estimation purposes, letting $N = r - 1$ would give an approximation to c_{rr}. For a self-adjoint system, c_{rr} reduces to zero; consequently, Eq. (44) would work well.

5. *Relationships Between Different Methods*

The generic formulas contain most of the current modal methods as a hierarchical structure according to the system type as well as the convergence rate. Table 2 exhibits how the various methods are obtained from the generic formulas. From Table 2, one knows that most of the available methods are limited to a special case of $\mu = 1$ and fixed shift value υ. By using the generic formulas, the limit is eliminated. This is the reason that two parameters are needed for deriving a generic formula.

6. *Summary of Computational Procedure and Comparison of Computational Cost*

Suppose that the first m eigenvalues and n_p system parameters are considered. The calculation parameters m_f, n_i, and x_ε are previously chosen according to predetermined ε and eigenvalues under consideration. Also, each $\partial \boldsymbol{A}/\partial p$ and $\partial \boldsymbol{B}/\partial p$ are previously obtained. By substituting the determined υ and μ [i.e., Eq. (35)] into the generic formulas, the principal procedure to calculate the right eigenvector derivatives is summarized in Algorithm 1.

Note that two nested loops are implicitly included in the procedure for m eigenvalues and n_p system parameters, respectively. The left eigenvector derivatives can be computed in a similar way.

Table 3 shows the approximate computational costs of several methods in the self-adjoint case. Only multiplications and divisions are considered. The flop count for $(\lambda_r \partial \boldsymbol{A}/\partial p + \partial \boldsymbol{B}/\partial p)\boldsymbol{\phi}_r$, $\boldsymbol{g}_r\boldsymbol{\phi}_r$, and $\partial \lambda_r/\partial p$ are not included because they are the same for all methods. The cost for calculating eigensolutions is also not included.

From Table 3, the cost of Fox and Kapoor's [1] method is the least for the same value of N_{tr}, among all of these methods. However, the method requires an excessive number of modes to guarantee accuracy. If a full set of modes is used, about $3.5N^3$ flop counts are needed for calculating eigenvectors [14]. For the same value of q, the cost of the presented method of $m_f = 1$ is slightly larger than the methods of Zeng and Zhang et al. . The difference of cost is because the former decomposes $(\lambda_r \boldsymbol{A} + 1/x_\varepsilon \boldsymbol{B})^{-1}$ according to different λ_r, whereas the latter decomposes $(d\boldsymbol{A} + \boldsymbol{B})^{-1}$ only once for a fixed d for all eigenvectors. Comparing with the presented method, the latter requires more modes and larger q for a predetermined accuracy.

At present, the method of $m_f \neq 1$ is implemented with matrix multiplication for each

$$(x_\varepsilon \lambda_r \boldsymbol{R}_H \boldsymbol{A}) \prod_{k=1}^{i} {}^{(n_{k-1}+1)}$$

and thus yields $\mathcal{O}(N^3)$ operation counts. However, once these matrices are obtained, the remains would be quickly computed.

Algorithm 1 Principal procedure of calculation of eigenvector derivative

Step 1:	LU decompose $\boldsymbol{B}^{-1}$, stored for later use
Step 2:	$\boldsymbol{g}_r$ and $\partial\lambda_r/\partial p$ [Eqs. (5) and (6)]
Step 3:	Compute $\boldsymbol{x}_q$ [Eq. (9)]
	a) $\boldsymbol{V}_0 = \boldsymbol{g}_r\boldsymbol{\phi}_r$
	b) $\boldsymbol{V}_k = (1/x_\varepsilon - 1)\boldsymbol{B}(1/x_\varepsilon\boldsymbol{B} + \lambda_r\boldsymbol{A})^{-1}\boldsymbol{V}_{k-1}$ for $k = 1, \ldots, q-1$ [Eqs. (9) and (29)]
	c) $\boldsymbol{x}_q = (1/x_\varepsilon\boldsymbol{B} + \lambda_r\boldsymbol{A})^{-1}\sum_{j=0}^{q-1}\boldsymbol{V}_j$
Step 4:	Compute α_{rr}, also $\alpha_{rk}(k < r)$, if necessary [Eq. (20) or Eq. (21) or Eq. (13)]
Step 5:	$\partial\boldsymbol{\phi}_r/\partial p$ [Eq. (12)]
Step 6:	Stop

Table 3 Comparison of computational costs

Method	Flop count
Nelson [4]	$m[N^3/3 - N/3 + (n_p + 2)N^2 + 2N]$
Fox and Kapoor [1]	$2mn_pNN_{\text{tr}}$
Wang [9] and Sutter et al. [7]	$N^3/3 - N/3 + mn_pN^2 + 2mn_pNN_{\text{tr}}$
Akgun [14]	$N^3/3 - N/3 + mn_p[(2q-1)N^2 + (q-1)N] + 2mn_pNN_{\text{tr}}$
Ma and Hagivara [15]	$N^3/3 - N/3 + N^2 + mn_pN^2 + 2mn_pNN_{\text{tr}}$
Zhang et al. [3]	$N^3/3 - N/3 + N^2 + mn_p[(2q-1)N^2 + (q-1)N] + 2mn_pNN_{\text{tr}}$
Presented ($m_f = 1$)	$N^3/3 - N/3 + mn_p[(2q(n_1+1)-1)N^2 + ([(2m+1)(n_1+1)+2m]q - 2m)N + ((2n_1+5)q - 2)m]$
($m_f \neq 1$)	$N^3/3 - N/3 + \left(2 + \sum_{i=1}^{m_f-1} n_i\right)N^3 + mN^2 + mN + mn_p\left(\left[\left(2 + \sum_{i=1}^{m_f} n_i\right)q - 1\right]N^2 + [2q(m+1)-1]N + 3mq\right)$

III. Numerical Examples

Two examples are used to illustrate the efficiency and validity of the presented method. The first example is on a non-self-adjoint system with zero eigenvalues, and the second one is on a system with hybrid damping, which demonstrates the ability to deal with complicated damping system. LAPACK routine `zggevx` is used to compute the eigenvalues and eigenvectors. Data types are complex double precision. Here, a conventional relative error definition $\delta = ||\partial\phi_{rE}/\partial p - \partial\phi_r/\partial p||_2/||\partial\phi_r/\partial p||_2$ other than Eq. (44) is used to directly measure the difference between the exact solutions and the results from the presented method.

A. Non-Self-Adjoint Example

Consider the following system

$$\boldsymbol{A} = \boldsymbol{I}, \qquad \boldsymbol{B} = \begin{bmatrix} -1 & p & p+1 & p & p-1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 1 & 0 & 1 & -1 & 0 & 1 \\ -1 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \end{bmatrix}$$

The matrix $\boldsymbol{B}$ is constructed so that the third and the fifth columns are the linear combinations of other independent columns. The eigenvalues for $p = 1.6$ are 0, 0, -0.6792, $0.6252 + 1.1519i$, $0.6252 - 1.1519i$, and -2.5712. When determining the eigensolutions of this system, a shift value of $1.5 \times 10^{-15} + 1.5 \times 10^{-15}i$ is used, because the original LAPACK routine `zggevx` gives unreasonable results for eigenvectors associated with zero eigenvalues. Note that the assumption of distinct eigenvalues is not a real limit to Eq. (3). In fact, Eq. (3) is always tenable for any nondefective system. For a system with repeated eigenvalues, equations following Eq. (3) are still applicable to other distinct eigenvalues, and so their corresponding eigenvector derivatives can directly be calculated by the presented method. Those associated with repeated eigenvalues can be determined with an existing method.

The left and right eigenvector derivatives associated with the third and the fourth eigenvalues with respect to parameter p are calculated here. The calculation parameters are $\varepsilon = 10^{-16}$, $m_f = 4$, $n_1 = 5$, $n_2 = 5$, $n_3 = 4$, $n_4 = 3$, $x_\varepsilon = 0.950119$, and $d = 0.02 - 0.02i$. Eigenvalue derivatives are -0.0380 and $0.1115 - 0.1559i$, respectively. Table 4 shows the eigenvector derivatives with eight-digit representation for the real and imaginary parts of each component, respectively. The results are compared with the exact solutions, which are obtained from the classical modal superposition method with full eigenmodes. Except the last digit is three in the imaginary part of the fourth component of the right eigenvector derivative of mode 4 (it is two in the exact solution), all of the other components are the same as the exact solution.

To investigate the convergence property of the presented method, cases of $q = 1$, 2, 4, 6, and 8 are also calculated. For $q = 1$, the maximum $\delta = 0.031$ and it occurs on the left eigenvector derivative of mode 3; $q = 2$, the maximum $\delta = 0.002$, on the right eigenvector derivative of mode 3; $q = 4$, the maximum $\delta = 9.22 \times 10^{-6}$, on the right eigenvector derivative of mode 3; $q = 6$, the maximum $\delta = 6.46 \times 10^{-8}$, on the right eigenvector derivative of mode 3; $q = 8$, the maximum $\delta = 1.57 \times 10^{-10}$, on the right eigenvector derivative of mode 4; and here, the results are the same as those listed in Table 4. It seems conservative to use the value of q determined by Eq. (40). By using Eq. (44), an approximate q could be estimated for predetermined δ. In addition, trials also verify that larger x_ε leads to faster convergence than does the smaller one; of course, either ε or n_i should change.

The example problem is also calculated with the parameters $\varepsilon = 1.0 \times 10^{-5}$, $m_f = 1$, $n_1 = 20$, $x_\varepsilon = 0.577969$, and $d = 0.02 - 0.02i$. The required value of q to have all relative errors less than 0.01 is six, and the maximum δ is 7.09×10^{-3} on the right eigenvector

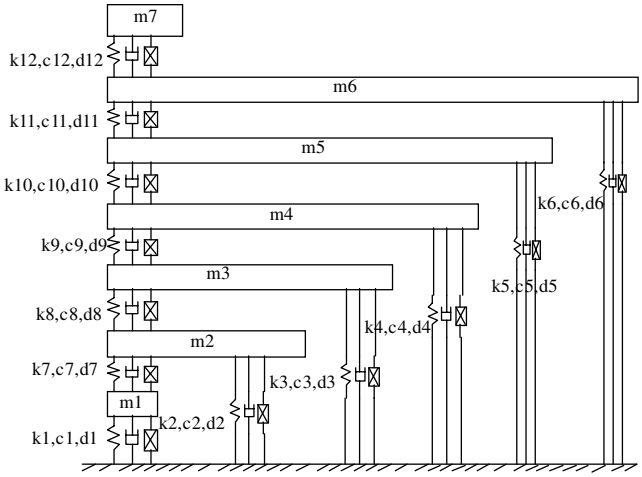

Fig. 1 Seven-DOF system with nonproportional viscous damping and hysteretic damping.

Table 4 Eigenvector derivatives associated with the 3rd and 4th eigenvalues

Mode no.	Left eigenvector derivatives	Right eigenvector derivatives
3	$0.72084070 \times 10^{-1}$	$0.10643555 \times 10^{-1}$
	0.12866039×10^{0}	$-0.25436928 \times 10^{-1}$
	$-0.36347875 \times 10^{0}$	$0.16909010 \times 10^{-1}$
	0.64909796×10^{0}	$-0.85279183 \times 10^{-2}$
	0.72118203×10^{0}	$0.14095235 \times 10^{-1}$
	$-0.49213914 \times 10^{0}$	$-0.19263556 \times 10^{-1}$
4	$0.30190013 \times 10^{-1} + 0.10711496 \times 10^{0}i$	$0.10692191 \times 10^{0} - 0.11414079 \times 10^{0}i$
	$-0.19559794 \times 10^{-1} + 0.34058984 \times 10^{0}i$	$-0.71707667 \times 10^{-1} - 0.21518481 \times 10^{0}i$
	$0.31216763 \times 10^{0} + 0.70934267 \times 10^{0}i$	$0.33268073 \times 10^{0} + 0.22389235 \times 10^{0}i$
	$-0.41703203 \times 10^{0} - 0.17417233 \times 10^{0}i$	$0.26097307 \times 10^{0} + 0.87075353 \times 10^{-2}i$
	$-0.38684202 \times 10^{0} - 0.67057372 \times 10^{-1}i$	$0.11110055 \times 10^{0} + 0.21874439 \times 10^{-1}i$
	$0.33172743 \times 10^{0} + 0.36875283 \times 10^{0}i$	$-0.11393727 \times 10^{0} + 0.27339613 \times 10^{0}i$

Table 5 Modal characteristics of the 7-DOF system

Mode no.	λ_r^1	λ_r^2	ω_r, Hz	ζ_r, %	η_r, %
1	$-0.9698 + 56.019i$	$-0.9527 - 56.019i$	8.917	1.716	0.0305
2	$-1.7353 + 96.083i$	$-1.7062 - 96.083i$	15.295	1.791	0.0303
3	$-2.6776 + 113.533i$	$-2.6431 - 113.533i$	18.074	2.343	0.0303
4	$-4.1224 + 135.410i$	$-4.0797 - 135.410i$	21.561	3.027	0.0315
5	$-5.1701 + 159.553i$	$-5.1207 - 159.553i$	25.407	3.223	0.0309
6	$-4.8955 + 173.796i$	$-4.8396 - 173.796i$	27.671	2.800	0.0322
7	$-5.5294 + 198.225i$	$-5.4696 - 198.225i$	31.561	2.773	0.0302

derivative of mode 3. With q up to 30, the results of mode 3 are identical with those listed in Table 4, and the maximum $\delta = 2.16 \times 10^{-6}$ occurs on the right eigenvector derivative of mode 4. Increasing q does not yield more accurate results of mode 4, because, as described in Eq. (31), the value of $|0.577969\lambda_4/\lambda_h|$ almost reaches the margin of error 1.0×10^{-5}.

The convergence process and the accuracy level reveal the good performance of the presented method.

B. Self-Adjoint Example

A seven-degree-of-freedom (7-DOF) mass-stiffness-damping system is shown in Fig. 1. There are two types of damping in the system: nonproportional viscous damping and hysteretic damping. Therefore, the modal characteristics of the system become complicated. System parameters are given as follows: m1 to m7 increasing from 10 to 16 kg, with an increment of 1 kg; k1 to k9 and k12 are 100 kN/m; k10 is 80 kN/m and k11 is 90 kN/m; c2 to c5, c7, c8, c10, and c11 are 30 N · s/m; c1 and c12 are 60 N · s/m; c6 is 80 N · s/m and c9 is 10 N · s/m; and d1 to d12 are 30 N/m.

The matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are given as

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{M} \\ \boldsymbol{M} & \boldsymbol{0} \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{K} + i\boldsymbol{D} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{M} \end{bmatrix}$$

where i denotes the complex identity. They are obtained through converting the standard dynamic equations in second-order form to state space. Because of complex coefficients matrix, eigenvalues and eigenvectors no longer appear as conjugate pairs, however, the imaginary parts of these quantities are still conjugate. Eigenvalues, natural frequencies, viscous damping ratios ζ_r, and damping loss factors η_r of the system are listed in Table 5.

Eigenvector derivatives of the first three modes with respect to stiffness coefficient k_6 are calculated. Here, $\alpha_{rr} = 0$ because $\partial \boldsymbol{A}/\partial k_6 = \boldsymbol{0}$. Four calculation cases are considered for the problem to verify the validity of the presented method. Only case 1 includes $a_{\mathrm{rk}}(k < r)$ to participate in the modal superposition, the others do not.

In case 1, the calculation parameters are $\varepsilon = 10^{-16}$, $m_f = 4$, $n_1 = 5$, $n_2 = 5$, $n_3 = 4$, $n_4 = 3$, and $x_\varepsilon = 0.950119$. The results associated with λ_r^1 $(r = 1, 2, 3)$ are listed in Table 6. Compared with the exact solutions, the maximum $\delta = 5.49 \times 10^{-11}$ occurs on $\partial \boldsymbol{\phi}_2 \phi_2/\partial k_6$, and each component is exactly identical with that of the exact solutions under eight-digit representation.

In case 2, the parameters are the same as in case 1. The maximum $\delta = 7.63 \times 10^{-9}$ occurs on $\partial \boldsymbol{\phi}_3/\partial k_6$. The last one or two digits of some components differ from that of the exact solutions under eight-digit representation.

Table 6 Derivatives of the first three eigenvectors with respect to k_6

$\partial \boldsymbol{\phi}_1/\partial k_6$	$\partial \boldsymbol{\phi}_2/\partial k_6$	$\partial \boldsymbol{\phi}_3/\partial k_6$
$-0.10166302 \times 10^{-9} + 0.85156856 \times 10^{-10}i$	$-0.17506143 \times 10^{-8} + 0.25894387 \times 10^{-8}i$	$0.36228648 \times 10^{-8} - 0.51534958 \times 10^{-8}i$
$0.49808231 \times 10^{-10} - 0.94443038 \times 10^{-10}i$	$-0.16845995 \times 10^{-8} + 0.23593428 \times 10^{-8}i$	$-0.49108629 \times 10^{-9} + 0.95408760 \times 10^{-9}i$
$0.66003449 \times 10^{-9} - 0.76161634 \times 10^{-9}i$	$-0.12522574 \times 10^{-8} + 0.16532642 \times 10^{-8}i$	$-0.69836181 \times 10^{-8} + 0.10135079\mathrm{E}-07i$
$0.26423565 \times 10^{-8} - 0.28593508 \times 10^{-8}i$	$-0.26696348 \times 10^{-9} + 0.28224762 \times 10^{-9}i$	$-0.10132331 \times 10^{-7} + 0.14425264 \times 10^{-7}i$
$0.98088906 \times 10^{-8} - 0.10409843 \times 10^{-7}i$	$0.14670521 \times 10^{-8} - 0.21927846 \times 10^{-8}i$	$-0.21700616 \times 10^{-8} + 0.40858085 \times 10^{-8}i$
$0.28543687 \times 10^{-7} - 0.30121457 \times 10^{-7}i$	$0.28301248 \times 10^{-8} - 0.41138135 \times 10^{-8}i$	$0.11464221 \times 10^{-7} - 0.15012053 \times 10^{-7}i$
$0.25965538 \times 10^{-8} - 0.17324789 \times 10^{-8}i$	$-0.68557279 \times 10^{-8} + 0.86698427 \times 10^{-8}i$	$-0.13192281 \times 10^{-7} + 0.16446954 \times 10^{-7}i$
$-0.25107692 \times 10^{-7} - 0.26336194 \times 10^{-7}i$	$-0.26547545 \times 10^{-6} - 0.18696563 \times 10^{-6}i$	$0.76998962 \times 10^{-6} + 0.57220144 \times 10^{-6}i$
$-0.28981968 \times 10^{-7} - 0.32013056 \times 10^{-7}i$	$-0.24467557 \times 10^{-6} - 0.18175984 \times 10^{-6}i$	$0.28671459 \times 10^{-7} + 0.49619302 \times 10^{-7}i$
$-0.28605717 \times 10^{-7} - 0.34157052 \times 10^{-7}i$	$-0.17855702 \times 10^{-6} - 0.14013043 \times 10^{-6}i$	$-0.11100812 \times 10^{-5} - 0.79890642 \times 10^{-6}i$
$0.70004240 \times 10^{-8} - 0.32784561 \times 10^{-8}i$	$-0.47287948 \times 10^{-7} - 0.42190198 \times 10^{-7}i$	$-0.17118126 \times 10^{-5} - 0.12695174 \times 10^{-5}i$
$0.20815836 \times 10^{-6} + 0.19213615 \times 10^{-6}i$	$0.19413565 \times 10^{-6} + 0.13399037 \times 10^{-6}i$	$-0.63294859 \times 10^{-6} - 0.38988928 \times 10^{-6}i$
$0.87247711 \times 10^{-6} + 0.84535413 \times 10^{-6}i$	$0.38668152 \times 10^{-6} + 0.27663924 \times 10^{-6}i$	$0.15871423 \times 10^{-5} + 0.12852116 \times 10^{-5}i$
$-0.14858960 \times 10^{-5} - 0.14267560 \times 10^{-5}i$	$-0.81379036 \times 10^{-6} - 0.66817291 \times 10^{-6}i$	$-0.17527743 \times 10^{-5} - 0.14852531 \times 10^{-5}i$

In case 3, ε decreases to be 10^{-12}, the corresponding x_ε becomes 0.962351, and the other parameters remain the same as those in case 1. The results are the same as those listed in Table 6. The maximum $\delta = 5.59 \times 10^{-11}$ occurs on $\partial \boldsymbol{\phi}_2 / \partial k_6$.

In case 4, the calculation is performed with changed parameters $m_f = 5$, $n_1 = 5$, $n_2 = 5$, $n_3 = 4$, $n_4 = 3$, $n_5 = 2$, and $x_\varepsilon = 0.983088$. The same results as those listed in Table 6 are obtained. The maximum $\delta = 6.69 \times 10^{-11}$ occurs on $\partial \boldsymbol{\phi}_2 / \partial k_6$. Here, the newly chosen m_f and n_i only increase by three multiplication operations.

The approach to directly solve the inverse matrix of $(\upsilon \boldsymbol{A} + \mu \boldsymbol{B})$, as described in a previous section, is also verified. Results consistent with the exact solutions are also obtained when q is up to 5 and $x_\varepsilon = 0.99$.

From the two numerical examples, one can find the following:

1) The decisive factor in the accuracy of eigenvector derivatives and convergence rate is x_ε.

2) It is not necessary to use excessively small ε. Highly accurate results could be obtained if ε provides sufficient accuracy for Eq. (29).

IV. Conclusions

An efficient modal superposition method is presented for calculation of eigenvector derivatives in either a self-adjoint system or a non-self-adjoint system. The available lower modes are used to decompose nonsingular matrices when derivatives of many eigenvectors are to be calculated, and a new algorithm is developed to speed up the convergence with the least computational cost. Only one eigenvalue and its corresponding left or right eigenvectors are required for modal superposition. The examples show that the method is efficient, and results comparable to the exact solutions can be obtained. The presented method is applicable to a complicated damping system and is appropriate for closed-mode cases.

Acknowledgments

This work was supported by the National Natural Science Foundation of the People's Republic of China (10272012) and New Century Excellent Talents at the University of Ministry of Education of the People's Republic of China (NCET-04-0169). The authors gratefully acknowledge the support of these organizations.

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